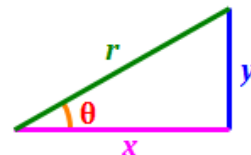


Introduction

What is Trigonometry?

The word “Trigonometry” comes from the Greek “trigonon” (meaning **triangle**) and “metron” (meaning **measure**). So, simply put, Trigonometry is the study of the measures of triangles. This includes the lengths of the sides, the measures of the angles and the relationships between the sides and angles.

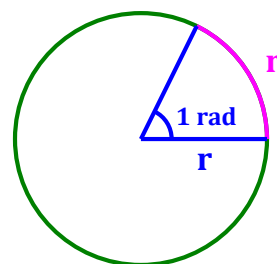


The modern approach to Trigonometry also deals with how right triangles interact with circles, especially the **Unit Circle**, i.e., a circle of radius 1. Although the basic concepts are simple, the applications of Trigonometry are far reaching, from cutting the required angles in kitchen tiles to determining the optimal trajectory for a rocket to reach the outer planets.

Radians and Degrees

Angles in Trigonometry can be measured in either radians or degrees:

- There are **360 degrees (i.e., 360°)** in one rotation around a circle. Although there are various accounts of how a circle came to have 360 degrees, most of these are based on the fact that early civilizations considered a complete year to have 360 days.
- There are **2π (~ 6.283) radians** in one rotation around a circle. The ancient Greeks defined π to be the ratio of the circumference of a circle to its diameter (i.e., $\pi = \frac{C}{d}$). Since the diameter is double the radius, the circumference is 2π times the radius (i.e., $C = 2\pi r$). One radian is the measure of the angle made from wrapping the radius of a circle along the circle's exterior.

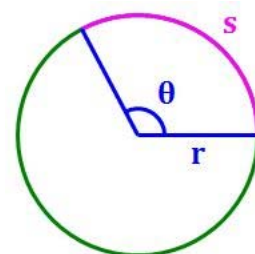


Measure of an Arc

One of the simplest and most basic formulas in Trigonometry provides the measure of an arc in terms of the radius of the circle, r , and the arc's central angle θ , expressed in radians. The formula is easily derived from the portion of the circumference subtended by θ .

Since there are 2π radians in one full rotation around the circle, the measure of an arc with central angle θ , expressed in radians, is:

$$S = C \cdot \left(\frac{\theta}{2\pi}\right) = 2\pi r \cdot \left(\frac{\theta}{2\pi}\right) = r\theta \quad \text{so} \quad S = r\theta$$



Angle Definitions

Basic Definitions

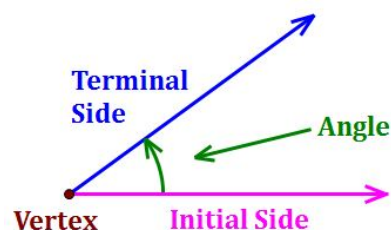
A few definitions relating to angles are useful when beginning the study of Trigonometry.

Angle: A measure of the space between rays with a common endpoint. An angle is typically measured by the amount of rotation required to get from its initial side to its terminal side.

Initial Side: The side of an angle from which its rotational measure begins.

Terminal Side: The side of an angle at which its rotational measure ends.

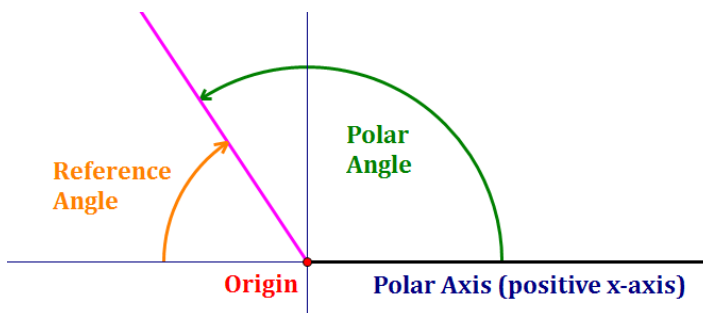
Vertex: The vertex of an angle is the common endpoint of the two rays that define the angle.



Definitions in the Cartesian (xy) Plane

When angles are graphed on a coordinate system (Rectangular or Polar), a number of additional terms are useful.

Standard Position: An angle is in **standard position** if its vertex is the origin (i.e., the point $(0, 0)$) and its initial side is the positive x -axis.



Polar Axis: The **Polar Axis** is the positive x -axis. It is the initial side of all angles in standard position.

Polar Angle: For an angle in standard position, its **polar angle** is the angle measured from the polar axis to its terminal side. If measured in a counter-clockwise direction, the polar angle is positive; if measured in a clockwise direction, the polar angle is negative.

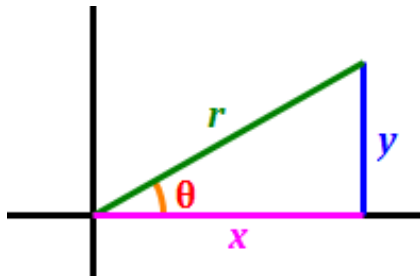
Reference Angle: For an angle in standard position, its **reference angle** is the angle between 0° and 90° measured from the x -axis (positive or negative) to its terminal side. The reference angle can be 0° ; it can be 90° ; it is never negative.

Coterminal Angle: Two angles are **coterminal** if they are in standard position and have the same terminal side. For example, angles of measure 50° and 410° are coterminal because 410° is one full rotation around the circle (i.e., 360°), plus 50° , so they have the same terminal side.

Quadrantal Angle: An angle in standard position is a **quadrantal angle** if its terminal side lies on either the x -axis or the y -axis.

Trigonometric Functions

Trigonometric Functions (on the x - and y -axes)



$$\sin \theta = \frac{y}{r}$$

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{x}{r}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{y}{x}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{x}{y}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{r}{x}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{r}{y}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

Pythagorean Identities

(for any angle θ)

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sec^2 \theta = 1 + \tan^2 \theta$$

$$\csc^2 \theta = 1 + \cot^2 \theta$$

Sine-Cosine Relationship

$$\sin \left(\theta + \frac{\pi}{2} \right) = \cos \theta$$

$$\sin \theta = \cos \left(\theta - \frac{\pi}{2} \right)$$

Key Angles

($180^\circ = \pi$ radians)

$$0^\circ = 0 \text{ radians}$$

$$30^\circ = \frac{\pi}{6} \text{ radians}$$

$$45^\circ = \frac{\pi}{4} \text{ radians}$$

$$60^\circ = \frac{\pi}{3} \text{ radians}$$

$$90^\circ = \frac{\pi}{2} \text{ radians}$$

Cofunctions (in Quadrant I)

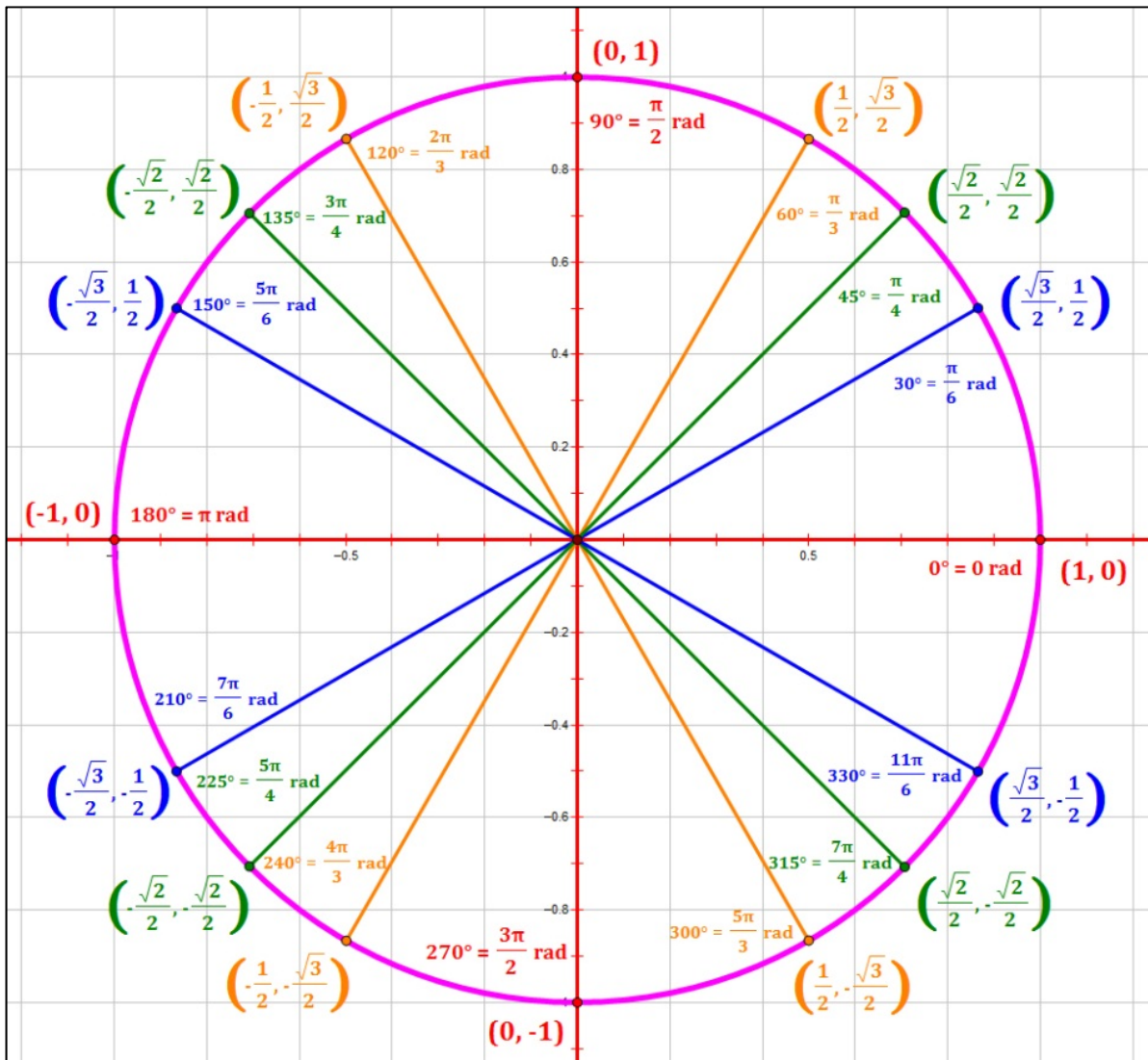
$$\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right) \quad \Leftrightarrow \quad \cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\tan \theta = \cot \left(\frac{\pi}{2} - \theta \right) \quad \Leftrightarrow \quad \cot \theta = \tan \left(\frac{\pi}{2} - \theta \right)$$

$$\sec \theta = \csc \left(\frac{\pi}{2} - \theta \right) \quad \Leftrightarrow \quad \csc \theta = \sec \left(\frac{\pi}{2} - \theta \right)$$

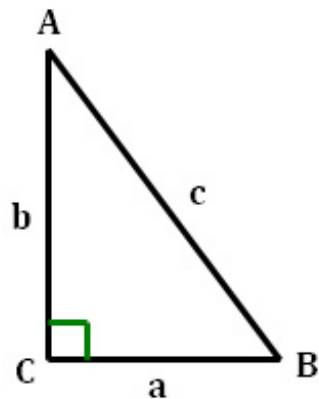
The Unit Circle

The **Unit Circle** diagram below provides x - and y -values on a circle of radius 1 at key angles. At any point on the unit circle, the x -coordinate is equal to the cosine of the angle and the y -coordinate is equal to the sine of the angle. Using this diagram, it is easy to identify the sines and cosines of angles that recur frequently in the study of Trigonometry.



Trigonometric Functions and Special Angles

Trigonometric Functions (Right Triangle)



SOH-CAH-TOA

$\sin = \frac{\text{opposite}}{\text{hypotenuse}}$
 $\cos = \frac{\text{adjacent}}{\text{hypotenuse}}$
 $\tan = \frac{\text{opposite}}{\text{adjacent}}$

$\sin A = \frac{a}{c}$
 $\cos A = \frac{b}{c}$
 $\tan A = \frac{a}{b}$

$\sin B = \frac{b}{c}$
 $\cos B = \frac{a}{c}$
 $\tan B = \frac{b}{a}$

Special Angles

Trig Functions of Special Angles (θ)				
Radians	Degrees	sin θ	cos θ	tan θ
0	0°	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{\sqrt{4}} = 0$
$\pi/6$	30°	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\pi/4$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}} = 1$
$\pi/3$	60°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$
$\pi/2$	90°	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	undefined

Note the patterns in the above table: In the sine column, the numbers 0 to 4 occur in sequence under the radical! The cosine column is the sine column reversed. Tangent = sine ÷ cosine.